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THE EIGENVALUES OF NON-SINGULAR TRANSFORMATIONS

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ABSTRACT
The eigenvalues of a non-singular conservative ergodic transformation of a separable measure space form a Borel subgroup of the circle of measure zero. We show that this is the only metric restriction on their size. However, the larger the eigenvalue group of the transformation, the "less recurrent" it is.

§1. Non-singular transformations

Let $(X, \mathcal{B}, m)$ be a separable probability space and $T: X \to X$ a non-singular, conservative ergodic transformation.

A measurable function $f: X \to \mathbf{C}$ is called an *eigenfunction* if there is a complex number $\lambda \in \mathbf{C}$ (*eigenvalue*) such that $f(Tx) = \lambda f(x)$ for m-a.e. $x \in X$. The conservativity of T implies that all eigenfunctions have constant modulus, and hence that all eigenvalues are unimodular. The ergodicity of T implies that eigenfunctions are unique up to constant multiplication.

We consider the collection of eigenvalues of T, which we denote by:

$$e(T) = \{s \in [0,1): \exists f_s : X \to \mathbf{T} \text{ measurable such that } f_s(Tx) = e^{2\pi i s} f_s(x) \text{ a.e.}\}.$$

(Here, $\mathbf{T} = \{\lambda \in \mathbf{C}: |\lambda| = 1\}$.) Clearly, $e(T)$ is a group under addition mod 1.

If T has a finite invariant measure $P \sim m$, then the eigenfunctions $\{f_\alpha\}_{\alpha \in e(T)}$ form an orthonormal system in $L^2(P)$ which is separable, so $e(T)$ *is countable.* (If T is allowed to be a finite measure preserving transformation of a non-separable measure space, then $e(T)$ can be any subgroup of $[0,1)$.)

It is known that, in general (when X is separable), $e(T)$ is a Borel subset of $[0,1)$ and there is a jointly measurable function (Lebesgue $\times$ Borel) $f: X \times e(T) \to \mathbf{T}$ so that

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$$f(Tx,s)=e^{2\pi is}f(x,s) \qquad \text{for every } s,x\in X_s$$

where $m(X\setminus X_s)=0$.

One way to prove this result is by considering the operator $Pg=\bar{g}g\circ T$ on unimodular measurable functions g. This operator is actually well defined on equivalence classes of constant (unimodular) multiples of such functions. On such objects, it is one-to-one by the ergodicity of T, and continuous with respect to convergence in measure of these classes. The collection of these classes g for which Pg is constant is closed and hence a complete separable metric space. The constants obtained are clearly eigenvalues of T. Thus $e(T)$, being the continuous, one-to-one image of a complete separable metric space, is a Borel set. The required function f is obtained by choosing a version of a suitable lifting of P^{-1} to the unimodular functions.

It now follows from the conservativity of T that $e(T)$ is a *weak Dirichlet set*, that is, whenever p is a probability measure charging $e(T)$ ($p(e(T))=1$):

$$\varliminf_{n\to\infty}\int_0^1|1-e^{2\pi ins}|^2dp(s)=0$$

(see [4], [9]). In particular, $e(T)$ has Lebesgue measure zero. Our first example shows that this is the only metric limitation on the size of $e(T)$: *for every gauge function $\rho:[0,1]\to[0,\infty]$ satisfying $\rho(t)\downarrow 0$, $\rho(t)/t\uparrow\infty$ as $t\downarrow 0$, there is a conservative ergodic transformation T of a separable measure space, with a σ-finite invariant measure, so that the ρ-Hausdorff measure of $e(T)$ is positive.*

However, transformations with large eigenvalue groups are forced to be "less recurrent". The term "less recurrent" refers to a concept introduced by Krengel ([8]).

Let $T:X\to X$ be a conservative ergodic transformation of $(X,\mathcal{B},m)$. Let $\hat{T}:L^1(X,m)\to L^1(X,m)$ be defined by

$$f\to dm_f=\int fdm\to dm_f\cdot T^{-1}/dm=\hat{T}f.$$

Then $\int_X\hat{T}fgdm=\int_X fg\cdot Tdm$, and the Chacon–Ornstein theorem states that:

$$\sum_{r=0}^{n-1}\hat{T}^rf(x)\Big/\sum_{r=0}^{n-1}\hat{T}^rg(x)\to\int_x fdm\Big/\int_x gdm \qquad \text{a.e.}$$

for f, $g\in L^1$, $\int gdm\neq 0$.

Using this, one can show ([8]) that *if $u_n\downarrow 0$ as $n\uparrow\infty$ then*:

$$\text{either}\quad \sum_{n=1}^{\infty}u_n\hat{T}^nf(x)=\infty \quad \text{a.e.}\quad \text{for every } f\geqq 0,\ \int_x fdm>0,$$

$$\text{or} \quad \sum_{n=1}^{\infty} u_n \hat{T}^n f(x) < \infty \quad \text{a.e.} \quad \text{for every } f \geqq 0, \quad \int_X f dm < \infty.$$

In the former case, T is called *u_n-conservative*, and in the latter case, T is called *u_n-dissipative.*

In case T has a σ-finite invariant measure $\mu \sim m$, then u_n-conservativity corresponds to:

$$\sum_{n=1}^{\infty} u_n f \cdot T^n = \infty \quad \text{a.e.} \quad \text{for every } f \geqq 0, \quad \int_X f d\mu > 0,$$

and u_n-dissipativity corresponds to:

$$\sum_{n=1}^{\infty} u_n f \cdot T^n < \infty \quad \text{a.e.} \quad \text{for every } f \geqq 0, \quad \int_X f d\mu < \infty.$$

It is known that T has a finite invariant measure iff T is u_n-conservative whenever $\sum_{n=1}^{\infty} u_n = \infty$.

It turns out that:

THEOREM 1. *If the Hausdorff dimension of $e(T)$ is larger than $\alpha \in (0,1)$, then T is $1/n^{1-\alpha}$-dissipative.*

Our second example shows that this proposition is sharp in the sense that:

For every $\alpha \in (0,1)$ there is an ergodic, $1/n^{1-\alpha}$-conservative transformation of a separable measure space with a σ-finite invariant measure whose eigenvalues have Hausdorff dimension α.

In §2, we prove Theorem 1, and Theorem 2 — a related result. In §3 we recall the definition of, and some facts about, dyadic towers over the adding machine. §4 is a lemma on Hausdorff measures (probably well known, but the author knows no reference). Examples are constructed in §5.

§2. Proof of Theorem 1

Under the assumption that the Hausdorff dimension of $e(T)$ is greater than $\alpha + \varepsilon$, we have, by a theorem of Frostman (see [3], [6]), that there is a probability measure p on $[0,1)$ satisfying $p(e(T)) = 1$, and $p((a,b)) \leqq M(b-a)^{\alpha+\varepsilon}$. This implies that:

$$I_p = \int_0^1 \int_0^1 \Phi_\alpha(t-s) dp(s) dp(t) < \infty$$

where

$$\Phi_\alpha(t)=\frac{1}{|(\sin \pi t)|^\alpha}\ .$$

Now Φ_α is convex on $(0,1)$ and so ([6]) $\hat{\Phi}_\alpha(n)>0$ ($\hat{\Phi}_\alpha(n)=\int_0^1 e^{2\pi ins}\,\Phi_\alpha(s)ds$) and this means that ([6]):

$$\sum_{n\in\mathbf{Z}}|\hat{p}(n)|^2\hat{\Phi}_\alpha(n)=I_p<\infty \qquad \left(\hat{p}(n)=\int_0^1 e^{2\pi ins}dp(s)\right).$$

It can be shown ([6]) that

$$\hat{\Phi}_\alpha(n)\sim\frac{\text{const}}{n^{1-\alpha}}$$

and so, recapitulating, we have

$$p(e(T))=1 \quad \text{and} \quad \sum_{n=1}^{\infty}|\hat{p}(n)|^2/n^{1-\alpha}<\infty.$$

Next, we set

$$G=\{g:[0,1]\to\mathbf{T}\ \text{measurable}\}$$

and

$$d_p(g,h)=\left\{\int_0^1|g(s)-h(s)|^2dp(s)\right\}^{1/2}.$$

Then (G,d_p) is a complete separable metric space, and a topological group under pointwise multiplication.

The above-mentioned function

$$f:X\times e(T)\to\mathbf{T}, \qquad f(Tx,s)=e^{2\pi is}f(x,s)$$

yields a function $\Pi:X\to G$ satisfying $\Pi(Tx)=g_0\Pi(x)$. (Here, $g_0(s)=e^{2\pi is}$ and $\Pi(x)(s)=f(x,s)$.)

Let $A(g,\varepsilon)=\{x\in X:d_p(\Pi(x),g)<\varepsilon\}$. Choose $h\in G$ so that $m(A(h,1/2))>0$. Suppose that $x\in G$, $n\geqq 1$ and $T^nx\in A(h,1/2)$. Then

$$d_p(h,\Pi(x))<\tfrac{1}{2}, \qquad d(h,\Pi(T^nx))<\tfrac{1}{2}$$

so

$$d_p(1,g_0^n)=d(\Pi(x),g_0^n\Pi(x))=d_p(\Pi(x),\Pi(T^nx))<1.$$

Now $d_p(1, g_0^n) = 2(1 - \operatorname{Re}\hat{p}(n)) < 1$ entails $|\hat{p}(n)| \geqq \operatorname{Re}\hat{p}(n) \geqq 1/2$. Rewriting this, we have that

$$1_{A(h,1/2)}(x) 1_{A(h,1/2)}(T^n x) \leqq 1_{(1/2,1]}(|\hat{p}(n)|).$$

Dividing by $n^{1-\alpha}$, summing over n and integrating over X we get:

$$\sum_{n=1}^{\infty} m(A(h, \tfrac{1}{2}) \cap T^{-n} A(h, \tfrac{1}{2})) \frac{1}{n^{1-\alpha}} \leqq \sum_{n=1}^{\infty} (1/n^{1-\alpha}) 1_{(1/2,1]}(|\hat{p}(n)|)$$

$$\leqq 4 \sum_{n=1}^{\infty} |\hat{p}(n)|^2 / n^{1-\alpha} < \infty.$$

In other words,

$$\sum_{n=1} (1/n^{1-\alpha}) \hat{T}^n 1_{A(h,1/2)} < \infty \quad \text{a.e.} \quad \text{on } A(h, 1/2)$$

and T is $1/n^{1-\alpha}$-dissipative. □

More generally suppose that $\Phi : (0,1) \to [0, \infty)$ is convex, and integrable on $(0,1)$, $\Phi(t) \uparrow \infty$ as $t \downarrow 0$. As remarked before, $\hat{\Phi}(n) \geqq 0$.

Let $E \subseteq [0,1]$ be a Borel set. One says ([6]) that the *Φ-capacity* of E is *positive* ($\Phi - \operatorname{cap} E > 0$) if there is a probability measure p on $[0,1]$ with $p(E) = 1$ and

$$\int_0^1 \int_0^1 \Phi(|t - s|) dp(s) dp(t) = \sum_{n=1}^{\infty} \hat{\Phi}(n) |\hat{p}(n)|^2 < \infty.$$

Any such measure p satisfies $p((a,b)) \leqq M/\Phi(|b-a|)$ and so the existence of such a measure ensures that the $1/\Phi$-Hausdorff measure of E is positive. The latter part of the proof of Theorem 1 can be used to prove:

THEOREM 2. *Suppose that Φ is such a function and $c_n \ll \hat{\Phi}(n)$, $c_n \downarrow 0$ as $n \uparrow \infty$. If $\Phi - \operatorname{cap} e(T) > 0$ then T is c_n-transient.*

This theorem has content when there is such a c_n, with $\Sigma c_n = \infty$, for example $\Phi = \Phi_\alpha$.

We conclude this section with some more examples of functions Φ for which Theorem 2 has content.

Suppose that $\Phi(x) = \Phi(1-x)$, $\Phi(x) \uparrow \infty$ as $x \downarrow 0$. If $t^3 \Phi''(t) \downarrow$ as $t \downarrow$ then $\hat{\Phi}(n) \sim C_n \downarrow 0$ as $n \uparrow \infty$.

If

$$\Phi''(x) \sim \frac{1}{x^\gamma} L(x) \qquad (x \downarrow 0)$$

where $L(x)$ is slowly varying ([2]) as $x \downarrow 0$ and $2 \leqq \gamma < 3$, then it can be shown using the theory of slowly varying functions that

$$\hat{\Phi}(n) \sim \text{const} \frac{1}{n^3} \Phi''\left(\frac{1}{n}\right) = \text{const} \frac{1}{n^{3-\gamma}} L\left(\frac{1}{n}\right)$$

which decreases in n.

Note that $\Phi''_\alpha(t) \sim \alpha(\alpha+1)/t^{\alpha+2}$ as $t \downarrow 0$, and hence $\hat{\Phi}(n) \sim \text{const}/n^{1-\alpha}$. If

$$\Phi_{\log}(t) = \log_e (1/|\sin \pi t|),$$

then

$$\Phi''_{\log}(t) \sim \frac{\text{const}}{t^2}, \quad \hat{\Phi}_{\log}(n) \sim \frac{\text{const}}{n} \qquad \text{as } n \to \infty.$$

§3. Dyadic towers over the adding machine

All of the examples to be constructed are *dyadic towers over the adding machine.*

Let $\Omega = \{0,1\}^{\mathbf{N}}$, and $\mathscr{A}$ be the σ-field generated by cylinders. Suppose $x \in \Omega$. Then $x = (\varepsilon_1(x), \varepsilon_2(x), \cdots)$. Define $l(x) = \min\{n \geqq 1 : \varepsilon_n(x) = 0\}$. Then, $x = (1 \cdots 1, 0, \varepsilon_{l(x)+1}(x), \cdots)$. Define

$$\tau(x) = (0 \cdots 0, 1, \varepsilon_{l(x)+1}(x), \cdots).$$

It is easy to see that for every $x \in \Omega$, $n \geqq 1$

$$\{(\varepsilon_1(\tau^k x), \varepsilon_2(\tau^k x), \cdots, \varepsilon_n(\tau^k x)) : 0 \leqq k \leqq 2^n - 1\} = \{0,1\}^n$$

and hence that τ preserves the measure $P = (\frac{1}{2}, \frac{1}{2})^{\mathbf{N}}$ and is ergodic (one proves constant limit in the ergodic theorem for functions depending on finitely many coordinates ε_n).

Recall from [1] that the *dyadic height function with heights* $\{\gamma(n)\}$ ($\gamma(n) \in \mathbf{N}, n \geqq 1$) is

$$\varphi(x) = \gamma(l(x)),$$

and that the dyadic tower over the dyadic adding machine $(X, \mathscr{B}, \mu, T)$ with height function φ is defined by

$$X = \{(x, n) : \varphi(x) \geqq n \geqq 1\},$$

$$\mathscr{B} = \bigvee_{n=1}^{\infty} (\mathscr{A} \cap [\varphi \geqq n], n),$$

$$m = \sum_{n=1}^{\infty} P_{|(\mathscr{A} \cap [\varphi \geqq n], n)},$$

$$T(x,n)=\begin{cases}(x,n+1) & \text{if } \varphi(x)\geqq n+1,\\ (\tau x,1) & \text{if } \varphi(x)=n.\end{cases}$$

Then ([7]), $(X,\mathcal{B},m,T)$ is a conservative measure preserving transformation, and ([5])

$$m(X)=\int_\Omega \varphi dP=\sum_{n=1}^{\infty}\frac{\gamma(n)}{2^n}.$$

Set $\beta(0)=\gamma(1)$ and $\beta(n)=\Sigma_{k=1}^{n}2^{n-k}\gamma(k)+\gamma(n+1)$ (called a *growth sequence* in [1]). Then

$$\gamma(n)=\beta(n-1)-\sum_{k=0}^{n-2}\beta(k)\qquad \text{for } n\geqq 2.$$

It will be convenient to determine the dyadic tower over the adding machine, T, by determining the sequence $\{\beta(n)\}_{n=0}^{\infty}\subseteq\mathbf{N}$ with $\beta(n)>\Sigma_{k=0}^{n-1}\beta(k)$. We then call T the *dyadic tower with growth sequence* $\{\beta(n)\}$. This is because ([1]):

$$\varphi_{2^n}\overset{\Delta}{=}\sum_{k=0}^{2^n-1}\varphi\circ\tau^k\geqq\beta(n-1)\quad\text{and}\quad P(\varphi_{2^n}=\beta(n))\geqq\tfrac{1}{2}.$$

Let $c(n)=\sup\{k\geqq 1:\beta(k)\leqq n\}$. It was shown in [1] that T is rationally ergodic with asymptotic type equivalent to $2^{c(n)}$. From this follows a property which we shall need:

there is an $A\in\mathcal{B}$, $m(A)>0$ *such that for every* $B\in\mathcal{B}$, $B\subseteq A$, $m(B)>0$:

$$\sum_{k=0}^{n-1}m(B\cap T^{-k}A)\overset{\cup}{\cap}2^{c(n)}.$$

PROPOSITION 3. *Let* $u_n\downarrow 0$ *as* $n\uparrow\infty$, *and* $\{\beta(n)\}$ *be a growth sequence. The dyadic tower over the adding machine is* u_n*-conservative iff* $\Sigma_{n=1}(u_n-u_{n+1})2^{c(n)}=\infty$.

PROOF. Let A be as in the above property:

$$\sum_{n=1}^{\infty}u_n 1_A\circ T^n=\sum_{n=1}^{\infty}(u_n-u_{n+1})\sum_{k=1}^{n}1_A\circ T^k\qquad\text{since } u_n\geqq u_{n+1}.$$

If T is u_n-conservative, then

$$\sum_{n=1}^{\infty}u_n 1_A\circ T^n=\infty\qquad\text{a.e.}$$

and

$$\infty=\int_A\sum_{n=1}^{\infty}u_n 1_A\circ T^n dm=\sum_{n=1}^{\infty}(u_n-u_{n+1})\sum_{k=1}^{n}m(A\cap T^{-k}A)$$

which implies that

$$\sum_{n=1}^{\infty}(u_n-u_{n+1})2^{c(n)}=\infty \qquad \text{since } \sum_{k=1}^{n} m(A\cap T^{-k}A) \ll 2^{c(n)}.$$

If T is u_n-dissipative then $\sum_{n=1}^{\infty} u_n 1_A\circ T^n<\infty$ a.e., and there is a set $B\subseteq A$, $B\in\mathcal{B}$, $m(B)>0$ such that

$$\infty>\int_B \sum_{n=1}^{\infty} u_n 1_A\circ T^n dm=\sum_{n=1}^{\infty}(u_n-u_{n+1})\sum_{k=1}^{n} m(B\cap T^{-k}A)$$

which implies that

$$\sum_{n=1}^{\infty}(u_n-u_{n+1})2^{c(n)}<\infty \qquad \text{since } \sum_{k=1}^{n} m(B\cap T^{-k}A) \gg 2^{c(n)}.$$ □

Proposition 4. *Let T be the dyadic tower over the adding machine with growth sequence $\beta(n)$:*

(a) *If $s\in[0,1]$ and $\sum_{n=1}^{\infty}|1-e^{2\pi i\beta(n)s}|<\infty$ then $s\in e(T)$.*

(b) *If $s\in e(T)$ then $e^{2\pi i\beta(n)s}\xrightarrow[n\to\infty]{}1$.*

Proof. (a) is proved in [4] (see also [1]). To see (b) note that $s\in e(T)$ iff there is a measurable function $f:\Omega\to\mathbf{T}$ with $f\circ\tau=e^{2\pi is\varphi}f$, whence $f\circ\tau^{2^n}=e^{2\pi is\varphi_{2^n}}f$ for $n\geqq 1$. It is easy to see that $g\circ\tau^{2^n}\xrightarrow[n\to\infty]{}g$ in measure for any $g:\Omega\to\mathbf{C}$ measurable (since $\varepsilon_k(\tau^{2^n}x)\xrightarrow[n\to\infty]{}\varepsilon_k(x)$). Hence $e^{2\pi is\varphi_{2^n}}\longrightarrow 1$ in measure, and, since $P(\varphi_{2^n}=\beta(n))\geqq\frac{1}{2}$, $e^{2\pi is\beta(n)}\xrightarrow[n\to\infty]{}1$. □

The examples T we construct will have growth sequences of the form $\beta(n)=2^{\delta(n)}$, $\delta(n)<\delta(n+1)$ where $\{\delta(n)\}_{n=1}^{\infty}=K\subseteq\mathbf{N}$,

$$K=\bigcup_{k=1}^{\infty}[n_k,n_k+m_k]\cap\mathbf{N} \qquad (n_{k+1}>n_k+m_k+k),$$

and we will write $T=T_K$, $\beta(n)=\beta_K(n)$ etc.

We have $c_K(n)=K\cap[1,[\log_2 n]]$ and hence $a_n(T_K)\,{\overset{\cup}{\cap}}\,2^{c_K(n)}$. Given a set $L\subseteq\mathbf{N}$, set

$$\Lambda(L)=\left\{s=\sum_{n=1}^{\infty}\frac{\varepsilon_n}{n}:\varepsilon_n=0,1 \text{ and } \varepsilon_n=0 \text{ for } n\in L\right\}.$$

We shall need to know when Hausdorff measures of $\Lambda(L)$ are positive, since, if

$$K=\bigcup_{k=1}^{\infty}[n_k,n_k+m_k]\cap\mathbf{N} \quad\text{and}\quad K_1=\bigcup_{k=1}^{\infty}[n_k,n_k+m_k+k]$$

where $n_{k+1} > n_k + m_k + k$, then:

$$\Lambda(K_1) \subseteq e(T_K).$$

This is because, for $s \in \Lambda(K_1)$:

$$((2^{n_k+j}s)) \leqq \frac{1}{2^{m_k+k-j}}, \qquad 0 \leqq j \leqq m_k$$

and so:

$$\sum_{n=1}^{\infty} |1 - e^{2\pi i \beta_K(n)s}| = 4 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} |\sin(\pi 2^{j+n_k}s)|$$

$$\leqq 4\pi \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \frac{1}{2^{k+m_k-j}} \leqq 4\pi$$

whence (Proposition 4 part (a)): $s \in e(T_K)$.

§4. A lemma on Hausdorff measure

LEMMA 5. *Suppose that $K \subseteq \mathbf{N}$ and $\rho : [0,1] \to [0,\infty]$, $\rho(t) \downarrow 0$ as $t \downarrow 0$ and $\rho(2t) \leqq M\rho(t)$ for $t > 0$.*

Then, if $A_{K,\rho} = \underline{\lim}_{n\to\infty} \rho(1/2^n) 2^{n-|K\cap[1,n]|}$,

$$\frac{A_{K,\rho}}{2M} \leqq H_\rho(\Lambda(K)) \leqq A_{K,\rho}.$$

In particular, the Hausdorff dimension of $\Lambda(K)$ is $1 - \overline{\lim}_{n\to\infty} (1/n)|K \cap [1,n]|$.

PROOF. For $n \geqq 1$, $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_n) \in \{0,1\}^n$, let

$$\sigma(\boldsymbol{\omega}) = \left\{ s = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k} ;\ \varepsilon_k = 0,1 \text{ and } \varepsilon_j = \omega_j \text{ for } 1 \leqq j \leqq n \right\}$$

$$= \left[\sum_{k=1}^{n} \frac{\omega_k}{2^k},\ \sum_{k=1}^{n} \frac{\omega_k}{2^k} + \frac{1}{2^n} \right]$$

(sets of this form are called dyadic intervals) and let

$$\Pi_n = \{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \{0,1\}^n,\ \sigma(\boldsymbol{\omega}) \cap \Lambda(K) \neq \varnothing\}.$$

From the definition of $\Lambda(K)$, we see that

$$\Pi_n = \{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \{0,1\}^n,\ \omega_k = 0 \text{ for } k \in K,\ 1 \leqq k \leqq n\},$$

and hence that $|\Pi_n| = 2^{n-|K\cap[1,n]|}$. Thus

$$H_\rho(\Lambda(K)) \leqq \underline{\lim}_{n\to\infty} \rho\left(\frac{1}{2^n}\right) 2^{n-|K\cap[1,n]|} = A_{K,\rho}.$$

Suppose $H_\rho(\Lambda(K)) = H < \infty$. Let $\varepsilon > 0$, then for every $n \geqq 1$ there are open intervals $I_1, I_2, \cdots, I_k, \cdots$ such that $\Lambda(K) \subseteq \bigcup_k I_k$, $\Sigma_{k=1}^{\infty} \rho(|I_k|) < H + \varepsilon$ and $|I_k| \leqq 1/2^{n+1}$, where $|I|$ denotes the length of I.

Now $\Lambda(K)$ is clearly compact and so $\exists N$ such that

$$\Lambda(K) \subseteq \bigcup_{k=1}^{N} I_k.$$

For any interval $I \subseteq [0,1)$ there exist dyadic intervals σ_1, σ_2 so that $I \subseteq \sigma_1 \cup \sigma_2$ and $|I| < |\sigma_1| = |\sigma_2| \leqq 2|I|$.

From this we deduce that there is a finite collection Π of dyadic intervals so that $\Lambda(K) \subseteq \bigcup_{\sigma \in \Pi} \sigma$, $|\sigma| \leqq 1/2^n$ and $\Sigma_{\sigma \in \Pi} \rho(|\sigma|) < 2M(H + \varepsilon)$.

Now, if σ and σ' are dyadic intervals and $|\sigma'| \leqq |\sigma|$ then

$$\text{either} \quad \sigma' \subseteq \sigma \quad \text{or} \quad \sigma' \cap \sigma = \emptyset.$$

Thus, Π can be chosen to be disjoint.

Next we set

$$\lambda(\sigma) = \log_2 \frac{1}{|\sigma|} \qquad (\lambda(\sigma(\omega_1, \cdots, \omega_n)) = n).$$

Let $\min\{\lambda(\sigma) : \sigma \in \Pi\} = q_\Pi \geqq n$ and $\max\{\lambda(\sigma) : \sigma \in \Pi\} = q_\Pi + r_\Pi$ $(r \geqq 0)$. If $r = 0$ then $\lambda(\sigma) = q$ $\forall \sigma \in \Pi$ and

$$\Pi = \Pi_q = \{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \{0,1\}^q,\ \omega_k = 0\ \forall k \in K\}.$$

So

$$2M(H + \varepsilon) > \sum_{\sigma \in \Pi_q} \rho(|\sigma|) = \rho\left(\frac{1}{2^q}\right) 2^{q - |K \cap [1,q]|} \qquad (q \geqq n).$$

In general, $r_\Pi \geqq 1$ and we next show that there is a $q' \geqq q$ so that

$$\sum_{\sigma \in \Pi_{q'}} \rho(|\sigma|) \leqq \sum_{\sigma \in \Pi} \rho(|\sigma|).$$

This is done in stages by showing that there is a collection of disjoint dyadic intervals Π' so that

$$\Lambda(K) \subseteq \bigcup_{\sigma \in \Pi'} \sigma, \quad q_{\Pi'} \geqq q_\Pi \quad \text{and} \quad r_{\Pi'} \leqq r_\Pi - 1.$$

Writing $\boldsymbol{\omega}(\sigma) = \boldsymbol{\omega}$ where $\sigma = \sigma(\boldsymbol{\omega})$, and $\lambda(\sigma) = \lambda(\boldsymbol{\omega}(\sigma))$ write $W = \{\boldsymbol{\omega}(\sigma) : \sigma \in \Pi\}$. For every $\boldsymbol{\omega} \in W$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_q, \omega_{q+1}, \cdots, \omega_{q+\nu})$ where $0 \leqq \nu \leqq r_\Pi$. Write $\boldsymbol{\omega} = (\boldsymbol{\theta}, \boldsymbol{\eta})$ where $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\omega}) = (\omega_1, \cdots, \omega_q)$ and

$$\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{\omega}) = \begin{cases} (\omega_{q+1}, \cdots, \omega_{q+\nu}) & \text{if } \nu \geqq 1, \\ \varnothing & \text{if } \nu = 0. \end{cases}$$

(We are introducing the conventions $(\boldsymbol{\omega}, \varnothing) = \boldsymbol{\omega}$, $\lambda(\varnothing) = 0$ and $\{0,1\}^0 = \{\varnothing\}$.) Clearly

$$\{\boldsymbol{\theta}(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \Pi\} = \Pi_q.$$

For $\boldsymbol{\theta} \in \Pi_q$ set $P_{\boldsymbol{\theta}} = \{\boldsymbol{\eta}(\boldsymbol{\omega}) : \boldsymbol{\theta}(\boldsymbol{\omega}) = \boldsymbol{\theta}\}$. Since Π is a disjoint collection, either $\boldsymbol{\theta} \in \Pi$ and $P_{\boldsymbol{\theta}} = \varnothing$ or $\lambda(\boldsymbol{\eta}) > 0$ for every $\boldsymbol{\eta} \in P_{\boldsymbol{\theta}}$.

Now,

$$\Pi = \bigcup_{\boldsymbol{\theta} \in \Pi_q} \{\sigma(\boldsymbol{\theta}, \boldsymbol{\eta}) : \boldsymbol{\eta} \in P_{\boldsymbol{\theta}}\}.$$

Moreover, for every $\boldsymbol{\theta} \in \Pi_q$,

$$\bigcup_{\substack{\sigma \in \Pi \\ \boldsymbol{\theta}(\boldsymbol{\omega}(\sigma)) = \boldsymbol{\theta}}} \sigma \supseteq \bigcup_{\substack{\sigma \in \Pi_{q+r} \\ \boldsymbol{\theta}(\boldsymbol{\omega}(\sigma)) = \boldsymbol{\theta}}} \sigma.$$

In other words, for every $\boldsymbol{\theta} \in \Pi_q$

$$\bigcup_{\boldsymbol{\eta} \in P_{\boldsymbol{\theta}}} \sigma(\boldsymbol{\eta}) \supseteq \{\boldsymbol{\varepsilon} \in \{0,1\}^{r_\Pi} : \varepsilon_k = 0 \text{ whenever } q + k \in K\}.$$

This shows that for every $\boldsymbol{\theta}_1 \in \Pi_q$:

$$\bigcup_{\boldsymbol{\theta} \in \Pi_q} \bigcup_{\boldsymbol{\eta} \in P_{\boldsymbol{\theta}_1}} \sigma(\boldsymbol{\theta}, \boldsymbol{\eta}) \supseteq \bigcup_{\sigma \in \Pi_{q+r}} \sigma \supseteq \Lambda(K).$$

We have that

$$\sum_{\sigma \in \Pi} \rho(|\sigma|) = \sum_{\boldsymbol{\theta} \in \Pi_q} \sum_{\boldsymbol{\eta} \in P_{\boldsymbol{\theta}}} \rho\left(\frac{1}{2^{q+\lambda(\boldsymbol{\eta})}}\right).$$

Choose $\boldsymbol{\theta}_0 \in \Pi_q$ so that $\sum_{\boldsymbol{\eta} \in P_{\boldsymbol{\theta}_0}} \rho(1/2^{q+\lambda(\boldsymbol{\eta})})$ is minimal. Set

$$\Pi' = \{(\boldsymbol{\theta}, \boldsymbol{\eta}) : \boldsymbol{\theta} \in \Pi_q, \boldsymbol{\eta} \in P_{\boldsymbol{\theta}_0}\}.$$

Then from the above:

$$\Lambda(K) \subseteq \bigcup_{\sigma \in \Pi'} \sigma \quad \text{and} \quad \sum_{\sigma \in \Pi'} \rho(|\sigma|) = \sum_{\boldsymbol{\theta} \in \Pi_q} \sum_{\boldsymbol{\eta} \in P_{\boldsymbol{\theta}_0}} \rho\left(\frac{1}{2^{q+\lambda(\boldsymbol{\eta})}}\right) \leqq \sum_{\sigma \in \Pi} \rho(|\sigma|)$$

by the choice of $\boldsymbol{\theta}_0$. If $P_{\boldsymbol{\theta}_0} = \{\varnothing\}$ ($\sigma(\boldsymbol{\theta}_0) \in \Pi$) then $\Pi' = \Pi_q$ and $r_{\Pi'} = 0$. If not, then $q_{\Pi'} \geqq q_\Pi + 1$, but $q_{\Pi'} + r_{\Pi'} \leqq q_\Pi + r_\Pi$ yielding $r_{\Pi'} \leqq r_\Pi - 1$.

A maximum of r_n such stages will show that there is a $q' \geqq n$ with

$$\rho\left(\frac{1}{2^{q'}}\right) 2^{q'-|K\cap[1,q']|} \leqq 2M(H+\varepsilon).$$

But for every $\varepsilon > 0$ and $n \geqq 1$ there is such a $q' \geqq n$ and so $A_{K,\rho} \leqq 2MH$.

We have shown that $A_{K,\rho} < \infty$ iff $H < \infty$ and in this case $A_{K,\rho}/2M \leqq H \leqq A_{K,\rho}$.

For $\rho_\alpha(t) = t^\alpha$, we have that $H_{\rho_\alpha}(\Lambda(K)) > 0$ iff $A_{K,\rho_\alpha} > 0$ iff

$$\varliminf_{n\to\infty}((1-\alpha)n - |K \cap [1,n]|) > -\infty,$$

whence the Hausdorff dimension of $\Lambda(K)$ is

$$\begin{aligned}\sup\{\alpha : H_{\rho_\alpha}(\Lambda(K)) > 0\} &= \sup\{\alpha : \varliminf_{n\to\infty}((1-\alpha)n - |K\cap[1,n]|) > -\infty\} \\ &= 1 - \varlimsup_{n\to\infty} \frac{1}{n}|K \cap [1,n]|. \end{aligned}$$

□

We are now in a position to present

§5. Examples

EXAMPLE 1. Given $\rho(t) \downarrow 0$, $\rho(t)/t \uparrow \infty$ there is a tower over the adding machine, T, so that

$$0 < H_\rho(e(T))$$

(this example is interesting when $\rho(t)$ is small, and $\rho(t)/t \uparrow \infty$ slowly).

To construct such an example, we find $n_k, m_k \geqq 1$, $n_{k+1} > n_k + m_k + k$ and set

$$K = \bigcup_{k=1}^{\infty} [n_k, n_k + m_k] \cap \mathbf{N}, \qquad K_1 = \bigcup_{k=1}^{\infty} [n_k, n_k + m_k + k] \cap \mathbf{N}$$

and $T = T_K$. We will have that $\Lambda(K_1) \subseteq e(T_K)$ and so it will suffice to choose n_k, m_k so that $H_\rho(\Lambda(K_1)) > 0$, or, equivalently (Lemma 5):

$$\varliminf_{n\to\infty} \rho\left(\frac{1}{2^n}\right) 2^{n-|K_1\cap[1,n]|} > 0.$$

To get this, we will have

$$|K_1 \cap [1,n]| \leqq n - R(n) \qquad \text{for every } n \geqq 1$$

where $R(n) = \log 1/\rho(1/2^n)$.

Since $\rho(t)/t \uparrow$ as $t \downarrow$, $n - R(n) \uparrow$ as $n \uparrow$, so this latter condition is equivalent to

$$\sum_{j=1}^{k} m_j + \frac{k(k+1)}{2} = |K_1 \cap [1, n_k + m_k + k]| \leqq n_k + m_k + k - R(n_k + m_k + k)$$

or

$$\sum_{j=1}^{k-1} m_j \leqq n_k - R(n_k + m_k + k) - \frac{k(k-1)}{2} \qquad \text{for every } k.$$

To construct sequences n_k, m_k, $n_{k+1} > n_k + m_k + k$ satisfying this choose m_1, n_1 arbitrarily. Since $n - R(n)$ increases, there is an $n_2 \geqq n_1 + m_1 + 1$ so that

$$m_1 \leqq n_2 - R(n_2) - 3.$$

Now $R(n)\uparrow$ and $n - R(n)\uparrow$ as $n \uparrow$ so $0 \leqq R(n+1) - R(n) \leqq 1$. Hence $R(n_2 + 3) - R(n_2) \leqq 3$, and setting $m_2 = 1$, we have

$$\begin{aligned} m_1 &\leqq n_2 - R(n_2 + m_2 + 1) + (R(n_2 + m_2 + 1) - R(n_2)) - 3 \\ &\leqq n_2 - R(n_2 + m_2 + 1) - 1. \end{aligned}$$

Next, suppose $n_1 \cdots n_{k-1}$, $m_1 \cdots m_{k-1}$ have been constructed. Choose $n_k > n_{k-1} + m_{k-1} + k$ so that

$$\sum_{j=1}^{k-1} m_j < n_k - R(n_k) - 2k - \frac{k(k-1)}{2}.$$

As before $R(n_k + 2k) - R(n_k) \leqq 2k$ so setting $m_k = k$, we obtain that

$$\begin{aligned} \sum_{j=1}^{k-1} m_j &\leqq n_k - R(n_k + m_k + k) + (R(n_k + m_k + k) - R(n_k)) - 2k - \frac{k(k-1)}{2} \\ &\leqq n_k - R(n_k + m_k + k) - \frac{k(k-1)}{2}. \end{aligned}$$

The set $K = \bigcup_{k=1}^{\infty} [n_k, n_k + m_k]$ having the required properties is thus constructed inductively.

EXAMPLE 2. Given $\alpha \in (0,1)$ there is a tower T over the adding machine so that $a_n(T) \gg n^{\alpha}$ (which implies by Proposition 3 that T is $1/n^{\alpha}$-conservative) and the Hausdorff dimension of $e(T)$ is $(1-\alpha)$.

Again, to construct the example, we will find n_k, m_k, $n_{k+1} > n_k + m_k + k$ defining K and K_1 as before and setting $T = T_K$. To get the required properties for T_K, we will arrange

$$|K \cap [1, n]| \geqq \alpha n - 1 \qquad \text{for } a_n(T) \gg n^{\alpha}$$

and

$$\overline{\lim_{n\to\infty}} \frac{|K_1 \cap [1,n]|}{n} \leqq \alpha \qquad \text{for Hausdorff dimension of } e(T) \text{ at least } (1-\alpha).$$

It follows from $a_n(T) \gg n^\alpha$ that T is $1/n^\alpha$-conservative, whence by Theorem 1, the Hausdorff dimension of $e(T)$ is at most $1-\alpha$.

To get n_k, m_k with the required properties, set $n_k = k^3$ and $m_k = [\alpha(3k^2+3k+1)]+1$. Then:

$$\alpha k^3 \leqq |K \cap [1,k^3]| < \alpha k^3 + k,$$

and it is easily checked that $|K \cap [1,n]| \geqq \alpha n$ for $n \geqq 1$.

Next, we see that

$$\begin{aligned}|K_1 \cap [1,k^3+m_k+k]| &= |K \cap [1,(k+1)^3]| + \frac{k(k+1)}{2} \\ &\leqq \alpha(k+1)^3 + \frac{(k+1)k}{2} \\ &\leqq \alpha(k^3+m_k+k) + M(k^3+m_k+k)^{2/3} \quad (\text{some } M<\infty)\end{aligned}$$

and it is easily checked that

$$|K_1 \cap [1,n]| \leqq \alpha n + Mn^{2/3}.$$

This completes the construction of Example 2.

We now discuss possibilities to improve Theorem 2. As mentioned before, if $\varphi:[0,1]\to(0,\infty)$ is convex, $\varphi(x)\uparrow\infty$ as $x\downarrow 0$ or $x\uparrow 1$, and $E\subseteq[0,1]$ is measurable, then $\varphi-\operatorname{cap} E>0$ implies that $H_{1/\varphi}(E)>0$. The author knows of no ergodic non-singular transformation of a separable measure space T with $H_{1/\varphi}(e(T))>0$ and T $\hat\varphi(n)$-conservative. It will follow from our concluding proposition that no dyadic tower over the adding machine of form T_K can have this property when $\varphi''(x)$ is regularly varying near zero with index $\gamma\in(-3,-2]$, and $\varphi(x)=\varphi(1-x)$.

Proposition 6. *Let $c_n\downarrow 0$, $\Sigma_{k=1}^n c_k = C(n)\uparrow\infty$ and $\rho(x)\sim 1/C([1/x])$. Let $K\subseteq\mathbf{N}$ and let T_K be the dyadic tower over the adding machine with growth sequence $\beta(n)=2^{k(n)}$ where $\{k(n)\}=K$.*

If $H_\rho(e(T_K))>0$ then T_K is c_n-dissipative.

(In case φ is convex, $\varphi(x)=\varphi(1-x)$ and $\varphi''(x)\sim L(x)/x^\gamma$ as $x\downarrow 0$ where $L(x)$ is slowly varying and $2\leqq\gamma<3$, one has that:

$$\hat\varphi(n)\sim c_n = \frac{1}{n^{3-\gamma}}L(1/n)$$

whence it follows that

$$\varphi(x) \sim \text{const}\, C([1/x])\ as\ x \downarrow 0.)$$

PROOF. First, note that there are integers $n_k \geqq 1$ and $m_k \geqq 0$ $(k \in \mathbf{N})$ such that $n_{k+1} \geqq n_k + m_k + 3$; $n_k, n_k + m_k \in K$, $K \subseteq \bigcup_{k=1}^{\infty} [n_k, n_k + m_k] \triangleq K_1$ and also such that if $n \in K$ and for some k, $n_k \leqq n \leqq n_k + m_k - 1$ then either $n + 1 \in K$, or $n + 2 \in K$.

Next, by Proposition 3, T_K is c_n-dissipative iff $\sum_{n=1}^{\infty} (c_n - c_{n+1}) 2^{c_K(n)} < \infty$. Here $c_K(n) = |K \cap [1, \log n]|$ and it follows that T_K is c_n-dissipative iff

$$S(K) \triangleq \sum_{n \in K} c_{2^n} 2^{K \cap [1,n]} < \infty.$$

We will show that, when $H_\rho(e(T_K)) > 0$, $S(K_1) < \infty$. This suffices because $S(K) \leqq S(K_1)$ (as $K \subseteq K_1$).

Set $K_2 = \bigcup_{k=1}^{\infty} [n_k, n_k + m_k + 2]$. Then

$$|K_2 \cap [1, n_k + m_k]| = |K_1 \cap [1, n_k + m_k]| + 2(k-1).$$

Set for $q \geqq 1$:

$$\Lambda_q = \left\{ s = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \in [0,1] \colon \varepsilon_n = 0,1 \text{ and for every } k \geqq q \colon \right.$$
$$\left. \varepsilon_{n_k+1} = \varepsilon_{n_k+2} = \cdots = \varepsilon_{n_k+m_k+2} \right\}.$$

By Proposition 4(b), if $s \in e(T_K)$ then

$$e^{2\pi i 2^n s} \xrightarrow[n \in K]{n \to \infty} 1$$

and so for some q: $\langle\!\langle (2^n s) \rangle\!\rangle < 1/2^5$ for every $n \geqq n_q$, $n \in K$. It follows from the construction of K_1 and K_2 that this entails $\langle\!\langle (2^n s) \rangle\!\rangle < 1/2^3$ for $n \geqq n_q$, $n \in K_2$ and this in turn implies that $s \in \Lambda_q$.

Hence $e(T_K) \subseteq \bigcup_{q=1} \Lambda_q$ and if $H_\rho(e(T_K)) > 0$ then, for some $q \geqq 1$: $H_\rho(\Lambda_q) > 0$.

For $n \geqq 1$ let Π_n denote the collection of dyadic intervals of length $1/2^n$ which intersect Λ_q. Since $H_\rho(\Lambda_q) > 0$, we have that $\inf_{n \geqq 1} \rho(1/2^n) |\Pi_n| > 0$. Now

$$|\Pi_{n_k+m_k}| \leqq 2^{n_k + m_k - |K_2 \cap [1, n_k + m_k]| + n_q + k}.$$

Whence (taking logarithms) there is a constant $M < \infty$ such that

$$|K_2 \cap [1, n_k + m_k]| \leqq n_k + m_k + k + \log_2 \rho(1/2^{n_k+m_k}) + M$$
$$= n_k + m_k + k - \log_2 C(2^{n_k+m_k}) + M.$$

Thus:

$$|K_1 \cap [1, n_k + m_k]| = |K_2 \cap [1, n_k + m_k]| - 2k + 2$$
$$\leqq n_k + m_k - k - \log_2 C(2^{n_k+m_k}) + M + 2.$$

But:

$$S(K_1) = \sum_{k=1}^{\infty} \sum_{n=0}^{m_k} c_{2^{n_k}+n} 2^{|K_1 \cap [1, n_k+n]|}$$

and now

$$\sum_{n=0}^{m_k} c_{2^{n_k}+n} 2^{|K_1 \cap [1, n_k+n]|} = \sum_{n=0}^{m_k} c_{2^{n_k}+n} 2^{|K_1 \cap [1, n_k+m_k]| - m_k + n}$$
$$\leqq 2^{M+2} \sum_{n=0}^{m_k} 2^{n_k+n} c_{2^{n_k}+n} / 2^k C(2^{n_k+m_k})$$
$$\leqq M'/2^k$$

which means $S(K_1) < \infty$. □

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